

Classical R -matrices for generalized $so(p, q)$ tops.

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An integrable deformation of the known integrable model of two interacting p -dimensional and q -dimensional spherical tops is considered. After reduction this system gives rise to the generalized Lagrange and the Kowalevski tops. The corresponding Lax matrices and classical r -matrices are calculated.

1 Introduction

The most common examples of classical R -matrices are associated with decompositions of Lie algebra into a direct sum of two Lie subalgebras [1]. According to [2] we can consider various "perturbations" of the standard decompositions of the loop algebras which may be associated with integrable deformations of the known integrable systems [3, 4]. In this note we consider integrable deformations of some tops associated with the Lie algebra $so(p, q)$ and calculate the corresponding R -matrices.

Let \mathfrak{g} be a self-dual Lie algebra, $\mathfrak{g}_+, \mathfrak{g}_- \subset \mathfrak{g}$ its two Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}_+ \dot{+} \mathfrak{g}_-$ as a linear space. Let P_+, P_- be the projection operators onto \mathfrak{g}_\pm parallel to the complimentary subalgebra; the operator

$$R = P_+ - P_- \tag{1.1}$$

is a classical R -matrix. If a is an intertwining operator

$$a[X, Y] = [aX, Y] = [X, aY] \tag{1.2}$$

for any $X, Y \in \mathcal{G}$, then $R_a = R \circ a$ is also a classical R -matrix [1]. So, one decomposition of \mathfrak{g} determines a family of R -brackets those orbits are related

$$L \mapsto a^{-1}L, \tag{1.3}$$

if a is invertible. For instance, such change of variables was used in [1] to connect Lax matrices for the Manakov top on $so(4)$ and for the Clebsch system on $e(3)$.

The most interesting class of examples is provided by loop algebras \mathcal{L}

$$\mathcal{L} = \mathfrak{g}[\lambda, \lambda^{-1}] = \left\{ X(\lambda) = \sum_i x_i \lambda^i, \quad x_i \in \mathfrak{g} \right\}. \tag{1.4}$$

The standard decomposition $\mathcal{L} = \mathcal{L}_+ \dot{+} \mathcal{L}_-$ is defined by the natural \mathbb{Z} -grading in powers of auxiliary variable λ (spectral parameter)

$$\mathcal{L}_+ = \bigoplus_{i \geq 0} \mathfrak{g}_i \lambda^i, \quad \mathcal{L}_- = \bigoplus_{i < 0} \mathfrak{g}_i \lambda^i \quad (1.5)$$

The R -bracket associated with decomposition (1.5) has a large collection of finite-dimensional Poisson subspaces

$$\mathcal{L}_{mn} = \bigoplus_{i=-m}^n \mathfrak{g}_i \lambda^i.$$

They are ad_R^* -invariant subspaces

$$ad_R^* X \cdot L = (ad_{\mathfrak{g}}^* X_+ \cdot L)_+ - (ad_{\mathfrak{g}}^* X_- \cdot L)_-, \quad X_{\pm} = P_{\pm} X, \quad L \in \mathcal{L}_{mn}, \quad (1.6)$$

which are invariant with respect to the Lax equation

$$L_t = [L, M], \quad M = P_+(dH(L)). \quad (1.7)$$

Intertwining operators in \mathcal{L} are multiplication operators by scalar Laurent polynomials. The crucial observation in [3, 4] is that to construct new Lax matrices by the rule (1.3) we can use the matrix Laurent polynomials a .

According to [2], if \mathfrak{g} is associative algebra, the same space \mathcal{L} (1.4) is a linear sum of \mathcal{L}_+ (1.5) and subalgebra \mathcal{L}_-^y

$$\mathcal{L} = \mathcal{L}_+ \dot{+} \mathcal{L}_-^y, \quad \mathcal{L}_-^y = \left\{ \sum_{i=1} x_i \lambda^i (1 - \lambda^{-1} y) \mid x_i \in \mathfrak{g}, \right\}, \quad (1.8)$$

defined by some fixed element $y \in \mathfrak{g}$. This new decomposition of \mathcal{L} defines another classical R -matrix (1.1)

$$R_y = P_+^y - P_-^y. \quad (1.9)$$

Subalgebra \mathcal{L}_-^y could be regarded as some "perturbation" of the standard subalgebra \mathcal{L}_- by element y . In this approach the finite-dimensional Poisson subspaces of the R_y -bracket (1.9)

$$\bigoplus_{i=-m}^n \mathfrak{g}_i \lambda^i (1 - \lambda^{-1} y) \quad \text{or} \quad (1 - \lambda^{-1} y)^{-1} \bigoplus_{i=-m}^n \mathfrak{g}_i \lambda^i,$$

are deformations of the known orbits \mathcal{L}_{mn} of R -bracket, which describe integrable deformations of the known integrable systems.

Our aim here is to show mapping of the Lax matrices

$$L = \lambda \mathbf{a} + \mathbf{l} + \lambda^{-1} \mathbf{s}, \quad M = \lambda \mathbf{b} \quad (1.10)$$

associated with Cartan type decomposition of $\mathfrak{g} = so(p, q)$ into the new Lax matrices associated with R_y -bracket

$$L_y = (1 - \lambda^{-1} y)^{-1} (\lambda \mathbf{a} + \mathbf{l}' + \lambda^{-1} \mathbf{s}), \quad M_y = M(1 - \lambda^{-1} y), \quad (1.11)$$

which describe new integrable deformations of $so(p, q)$ tops.

2 Interacting spherical tops on $so(p, q)$ algebra

Let $G = SO(p, q)$ is the group of pseudo-orthogonal matrices with signature (p, q) , $p \geq q$. The Lie algebra $so(p, q)$ consists of all the $(p + q) \times (p + q)$ matrices satisfying

$$X^T = -J X J,$$

where $J = \text{diag}(1, \dots, 1; -1, \dots, -1)$, $\text{tr} J = p - q$ and T means a matrix transposition.

In the natural (p, q) -block notation an element $X \in so(p, q)$ has the form

$$X = \begin{pmatrix} \ell & s \\ s^T & m \end{pmatrix}, \quad (2.1)$$

where $\ell^T = -\ell$, $m^T = -m$ are $p \times p$ and $q \times q$ matrices, s is an arbitrary $p \times q$ matrix.

The Cartan involution is given by $\sigma X = -X^T$ and $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ is the corresponding Cartan decomposition. The maximal compact subalgebra $\mathfrak{f} = so(p) \oplus so(q)$ consists of matrices X with $s = 0$

$$\mathfrak{f} = so(p) \oplus so(q) = \left\{ \begin{pmatrix} \ell & 0 \\ 0 & m \end{pmatrix} \right\}$$

The subspace \mathfrak{p} consists of matrices X with $\ell = 0$, $m = 0$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & s \\ s^T & 0 \end{pmatrix} \right\}.$$

The pairing between $so(p, q)$ and $so(p, q)^*$ is given by invariant inner product $(X, Y) = -\frac{1}{2} \text{tr} X Y$ which is positive definite on \mathfrak{f} .

We extend the involution σ to the loop algebra $\mathcal{L}(\mathfrak{g}, \sigma)$ by setting $(\sigma X)(\lambda) = \sigma(X(-\lambda))$. By definition, the twisted loop algebra $\mathcal{L}(so(p, q), \sigma)$ consists of matrices $X(\lambda)$ such that

$$X(\lambda) = -X^T(-\lambda). \quad (2.2)$$

The pairing between $\mathcal{L}(\mathfrak{g}, \sigma)$ and $\mathcal{L}(\mathfrak{g}, \sigma)$ is given by $\langle X, Y \rangle = \text{Res } \lambda^{-1}(X, Y)$.

2.1 Quadratic Hamiltonian

Let $k \in SO(p)$, $r \in SO(q)$ and $\ell \in so(p)$, $m \in so(q)$ be the configuration and momentum variables on the phase space $T^*SO(p) \times T^*SO(q)$. All the non-zero Lie-Poisson brackets are equal to

$$\begin{aligned} \{\ell_{ij}, \ell_{mn}\} &= \delta_{in} \ell_{jm} + \delta_{jm} \ell_{in} - \delta_{im} \ell_{jn} - \delta_{jn} \ell_{im}, \\ \{\ell_{ij}, k_{nm}\} &= \delta_{jm} k_{ni} - \delta_{im} k_{nj}, \\ \{m_{ij}, m_{kn}\} &= \delta_{in} m_{jk} + \delta_{jk} m_{in} - \delta_{ik} m_{jn} - \delta_{jn} m_{ik} \\ \{m_{ij}, r_{nk}\} &= \delta_{jk} r_{ni} - \delta_{ik} r_{nj}. \end{aligned} \quad (2.3)$$

The Poisson mapping from $T^*SO(p) \times T^*SO(q)$ into $\mathcal{L}(so(p, q), \sigma)$ is defined by the following theorem [1].

Theorem 1 (*Reyman, Semenov-Tian-Shansky*)
The spectral invariants of the Lax matrix

$$\begin{aligned} L &= \lambda \mathbf{a} + \mathbf{l} + \lambda^{-1} \mathbf{s} \\ &= \begin{pmatrix} 0 & k A r^T \\ r A^T k^T & 0 \end{pmatrix} \lambda - \begin{pmatrix} k \ell k^T & 0 \\ 0 & r m r^T \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix}, \end{aligned} \quad (2.4)$$

are in involution with respect to the canonical Poisson brackets (2.3) on $T^*SO(p) \times T^*SO(q)$. Here A and F are constant $p \times q$ matrices.

We refer the reader to [1] for a complete proof of this theorem which uses that $L(\lambda)$ (2.4) is an orbit of classical R -matrix (1.1) associated with a standard decomposition of the twisted loop algebra $\mathcal{L}(so(p, q), \sigma)$.

Here we present an elementary proof using a tensor form of the R -bracket which is more familiar in the inverse scattering method [1]. According to [6] functions $\phi_k(L) = \text{tr } L^k$, $k \geq 2$ are in the involution if and only if matrix $L(\lambda)$ satisfies relation

$$\left\{ \overset{1}{L}(\lambda), \overset{2}{L}(\mu) \right\} = \left[r_{12}(\lambda, \mu), \overset{1}{L}(\lambda) \right] - \left[r_{21}(\lambda, \mu), \overset{2}{L}(\mu) \right]. \quad (2.5)$$

Here we used the standard notations

$$\overset{1}{L}(\lambda) = L(\lambda) \otimes 1, \quad \overset{2}{L}(\mu) = 1 \otimes L(\mu),$$

and matrices r_{12} , r_{21} are kernels of the operators R and R^* such that

$$r_{21}(\lambda, \mu) = \Pi r_{12}(\mu, \lambda) \Pi,$$

where Π is a permutation operator $\Pi X \otimes Y = Y \otimes X \Pi$.

One checks without difficulty that $L(\lambda)$ (2.4) satisfies (2.5) with the following kernel of the R -matrix

$$r_{12}(\lambda, \mu) = -\frac{\lambda^2}{\lambda^2 - \mu^2} P_{\mathfrak{f}} + \frac{\lambda \mu}{\lambda^2 - \mu^2} P_{\mathfrak{p}}, \quad (2.6)$$

where $P_{\mathfrak{f}}$ and $P_{\mathfrak{p}}$ are kernels of the standard Casimir operators acting in the orthogonal subspaces \mathfrak{f} and \mathfrak{p} respectively. They are equal to

$$P_{\mathfrak{f}} = \sum_{\alpha=1}^{\frac{p(p-1)+q(q-1)}{2}} Z_{\alpha} \otimes Z_{\alpha}, \quad P_{\mathfrak{p}} = \sum_{\beta=1}^{pq} S_{\beta} \otimes S_{\beta},$$

where antisymmetric Z_{α} and symmetric S_{β} matrices form two orthonormal bases in \mathfrak{f} and \mathfrak{p} . For instance we can use

$$Z_1 = \left(\begin{array}{cccc|c} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline 0 & & & & 0 \end{array} \right), \quad Z_2 = \left(\begin{array}{cccc|c} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \\ -1 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline 0 & & & & 0 \end{array} \right), \dots,$$

and

$$S_1 = \left(\begin{array}{ccc|ccc} 0 & & & 1 & 0 & \cdots \\ & & & \vdots & \vdots & \ddots \\ \hline 1 & 0 & \cdots & & & \\ \vdots & \vdots & \ddots & & 0 & \end{array} \right), \quad S_2 = \left(\begin{array}{ccc|ccc} 0 & & & 0 & 1 & \cdots \\ & & & \vdots & \vdots & \ddots \\ \hline 0 & 0 & \cdots & & & \\ 1 & \vdots & \ddots & & 0 & \end{array} \right), \dots$$

Corollary 1 *The equations of motion generated by the spectral invariants of the matrix $L(\lambda)$ (2.4) are Hamiltonian equations with respect to canonical brackets (2.3), which give rise to Lax equation (1.7).*

Let A be the truncated diagonal matrix $A = a_i \delta_{ij}$. The spectral invariants $\phi_k(L)$ of $L(\lambda)$ (2.4) may be combined in the following Hamiltonian

$$\begin{aligned} H = & \sum_{i < j \leq q} \frac{a_i b_i - a_j b_j}{a_i^2 - a_j^2} (\ell_{ij}^2 + m_{ij}^2) - 2 \sum_{i < j \leq q} \frac{a_i b_j - a_j b_i}{a_i^2 - a_j^2} \ell_{ij} m_{ij} + \\ & + \sum_{i=q+1}^p \sum_{j=1}^q \frac{b_j}{a_j} \ell_{ij}^2 + \gamma \sum_{i,j=q+1}^p \ell_{ij}^2 - 2 \sum_{i,j,s} b_i k_{ij} F_{js} r_{is}, \end{aligned} \quad (2.7)$$

which formally describes an interaction of p -dimensional and q -dimensional spherical tops. Here $a_i, b_i, i = 1 \dots q$ and γ are arbitrary parameters [1]. In this case the second Lax matrix M in (1.7) is equal to

$$M(\lambda) = P_+(dH(L)) = - \begin{pmatrix} 0 & k B r^T \\ r B^T k^T & 0 \end{pmatrix} \lambda, \quad (2.8)$$

where diagonal matrix $B = b_i \delta_{ij}$ includes parameters b_i .

2.2 Integrable deformations

Let us introduce matrix

$$y = c \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix}, \quad c \in \mathbb{C}, \quad (2.9)$$

which is a generic solution of the following equations

$$y X s = \sigma(y X s), \quad \forall X \in \mathfrak{g}, \quad s = \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix} \in \mathfrak{p}. \quad (2.10)$$

This solution is parametrized by an arbitrary numerical parameter c .

The matrix y (2.9), $y \in so(p+q) = \mathfrak{f} + i\mathfrak{p}$, does not belong to the algebra $so(p, q)$ (2.1). So, to consider deformations \mathcal{L}_-^y (1.8) of \mathcal{L}_- we have to embed initial algebra \mathfrak{g} and the element y into some algebra $\tilde{\mathfrak{g}}$ and only then to discuss mapping of the known orbit (1.10) of R -bracket (1.5) into the orbit (1.11) of R_y -bracket (1.9) defined in $\mathcal{L}(\tilde{\mathfrak{g}})^*$.

Therefore, for brevity, we shall consider kernels $r(\lambda, \mu)$ and $r^y(\lambda, \mu)$ instead of the corresponding operators R (1.1,1.5) and R_y (1.9,1.8).

Proposition 1 *The spectral invariants of the Lax matrix*

$$L_y = (1 - \lambda^{-1}y)^{-1}(\lambda \mathbf{a} + \mathbf{l}' + \lambda^{-1} \mathbf{s}), \quad (2.11)$$

where

$$\mathbf{l}' = \mathbf{l} - \frac{1}{2} \left(y \mathbf{a} + \mathbf{a} y \right), \quad (2.12)$$

are in involution with respect to the canonical Poisson brackets (2.3) on $T^*SO(p) \times T^*SO(q)$.

To proof this proposition we have to check equation (2.5) with the kernel of the operator R_y

$$r_{12}^y(\lambda, \mu) = \left[1 \otimes (1 - \mu^{-1}y)^{-1} \right] r_{12}(\lambda, \mu) \left[(1 - \lambda^{-1}y) \otimes 1 \right]. \quad (2.13)$$

by using properties of the element y (2.10) and definition of matrices $r_{ij}(\lambda, \mu)$ (2.6).

Corollary 2 *The equations of motion generated by the spectral invariants of the matrix L_y (2.11) are Hamiltonian equations with respect to canonical brackets (2.3). They give rise to Lax equation (1.7), where*

$$M_y = P_+^y(dH(L)) = M(1 - \lambda^{-1}y). \quad (2.14)$$

The Lax equation (1.7) for L_y (2.11) may be rewritten as a Lax triad

$$\frac{d}{dt} L_i = L_i M L_2 - L_2 M L_i, \quad i = 1, 2 \quad (2.15)$$

on the pair of matrices

$$L_1 = L - \frac{1}{2}(y \mathbf{a} + \mathbf{a} y) \quad \text{and} \quad L_2 = 1 - \lambda^{-1}y,$$

entering in the definition of L_y (2.11).

For future reference we present two equivalent forms $H_{1,2}^y$ of the deformed Hamiltonian H^y using additional canonical transformations of the phase space

$$\mathbf{l} \rightarrow \mathbf{l} \pm \frac{c}{2} (\mathbf{s} \mathbf{a} - \mathbf{a} \mathbf{s}). \quad (2.16)$$

Compositions of these canonical transformations with noncanonical map (2.12)

$$\mathbf{l} \rightarrow \mathbf{l}_{1,2} = \mathbf{l} - \frac{1}{2} \left((y \mp c \mathbf{s}) \mathbf{a} + \mathbf{a} (y \pm c \mathbf{s}) \right)$$

act either in the subalgebra $so(p)$

$$\ell \rightarrow \ell_1 = \ell - c \left(k^T F r A^T - A r^T F^T k \right), \quad (2.17)$$

either in the subalgebra $so(q)$

$$m \rightarrow m_1 = m - c \left(r A^T k^T F - F^T k A r^T \right). \quad (2.18)$$

The corresponding perturbations of the initial Hamiltonian H (2.7) depend either on ℓ variables

$$H_1 = H - c \cdot \text{tr}(k \ell B r^T F^T) + \frac{c^2}{2} \text{tr}(B^T A r F^T F r^T) \quad (2.19)$$

either on m variables

$$H_2 = H + c \cdot \text{tr}(r m B^T k^T F) + \frac{c^2}{2} \text{tr}(B A^T k^T F F^T k). \quad (2.20)$$

The corresponding Lax matrices are equal to

$$L_y^{(1)} = (1 - \lambda^{-1}y)^{-1} \left[L(\lambda) + c \begin{pmatrix} Fr A^T k^T - k A r^T F^T & 0 \\ 0 & 0 \end{pmatrix} \right] \quad (2.21)$$

or

$$L_y^{(2)} = (1 - \lambda^{-1}y)^{-1} \left[L(\lambda) - c \begin{pmatrix} 0 & 0 \\ 0 & Fr A^T k^T - k A r^T F^T \end{pmatrix} \right]. \quad (2.22)$$

We have discussed so far Lax matrices for the tops in the stationary reference frame, i.e. used Euler-Lagrange description of motion. According to [1], we can go over to the Lax matrices in the frame moving with the body which amounts to the gauge transformation

$$\begin{aligned} \tilde{L}(\lambda) &= g^T L g = \lambda \tilde{\mathbf{a}} + \tilde{\mathbf{l}} + \lambda^{-1} \tilde{\mathbf{s}} = \\ &= \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \lambda - \begin{pmatrix} \ell & 0 \\ 0 & m \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & k^T F r \\ r^T F^T k & 0 \end{pmatrix}, \\ \tilde{M}(\lambda) &= g^T M(\lambda) g + w = - \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \lambda + g^T \frac{d}{dt} g. \end{aligned} \quad (2.23)$$

Here $g = \begin{pmatrix} k & 0 \\ 0 & r \end{pmatrix}$ and $w = g^T \dot{g}$ is the block-matrix of angular velocities, related to ℓ and m by a linear transformations.

Proposition 2 *The Lax matrix $\tilde{L}(\lambda)$ (2.23) associated with the Euler-Poisson description of the motion satisfies equation (2.5) with the following matrices $\tilde{r}_{ij}(\lambda, \mu)$*

$$\tilde{r}_{12}(\lambda, \mu) = -r_{12}(\lambda^{-1}, \mu^{-1}), \quad \tilde{r}_{21}(\lambda, \mu) = -r_{21}(\lambda^{-1}, \mu^{-1}), \quad (2.24)$$

which are obtained from $r_{12}(\lambda, \mu)$ (2.6) by change of the spectral parameters.

Using the similar gauge transformation to the matrices $L_y(\lambda)$ (2.11) and $M_y(\lambda)$ (2.14) one gets

$$\begin{aligned} \tilde{L}_y(\lambda) &= g^T L_y g = (1 - \lambda^{-1} \tilde{y})^{-1} \left(\tilde{L}(\lambda) - \frac{1}{2} (\tilde{y} \tilde{\mathbf{a}} + \tilde{\mathbf{a}} \tilde{y}) \right), \\ \tilde{M}_y(\lambda) &= g^T M_y(\lambda) g + g^T \frac{d}{dt} g = \tilde{M} (1 - \lambda^{-1} \tilde{y}). \end{aligned} \quad (2.25)$$

Here matrices $w_y = g^T \dot{g}$ are calculated with respect to cubic Hamiltonian H^y and, therefore, it depends on all the dynamical variables. As above (2.15), a pair of matrices

$$L_1 = \tilde{L} + \frac{1}{2}(\tilde{y}\tilde{\mathbf{a}} + \tilde{\mathbf{a}}\tilde{y}) \quad \text{and} \quad L_2 = 1 - \lambda^{-1}\tilde{y},$$

entering in the definition \tilde{L}_y (2.25), satisfy to the Lax triad

$$\frac{d}{dt}L_i(\lambda) = L_i(\lambda)\tilde{M}(\lambda) + \tilde{M}^T(-\lambda)L_i(\lambda), \quad i = 1, 2. \quad (2.26)$$

These equations (2.26) have a more complicated structure with respect to Lax triad in the rest frame (2.15)

In the body frame the matrix

$$\tilde{y} = g^T y g = c \begin{pmatrix} 0 & k^T F r \\ -r^T F^T k & 0 \end{pmatrix}$$

depends on the dynamical variables and, therefore, the kernel (2.13) of the corresponding operator R_y depends on these variables too.

However in the moving frame we can construct another Lax matrix such that the kernel of the corresponding R -matrix does not depend on dynamical variables. Namely, using matrix

$$\tilde{z} = c \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \quad (2.27)$$

such that

$$\tilde{z} X \mathbf{a} = \sigma(\tilde{z} X \mathbf{a}), \quad \forall X \in \mathfrak{g}, \quad \mathbf{a} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \in \mathfrak{p}, \quad (2.28)$$

we can consider perturbation of the Lax matrix $\tilde{L}(\lambda)$ (2.23) by \tilde{z}

$$\tilde{L}_z(\lambda) = (1 - \lambda\tilde{z})^{-1} \left(\tilde{L}(\lambda) - \frac{1}{2}(\tilde{z}\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\tilde{z}) \right). \quad (2.29)$$

Proposition 3 *The Lax matrix $\tilde{L}_z(\lambda)$ (2.29) satisfies equation (2.5) with the following numerical r -matrix*

$$\tilde{r}_{12}^z(\lambda, \mu) = \left[1 \otimes (1 - \lambda\tilde{z})^{-1} \right] \tilde{r}_{12}(\lambda, \mu) \left[(1 - \mu\tilde{z}) \otimes 1 \right]. \quad (2.30)$$

The proof is straightforward.

Applying inverse gauge transformation to $\tilde{L}_z(\lambda)$ one gets another Lax matrix $L_z(\lambda)$ in the stationary frame

$$L_z(\lambda) = g L_z g^T = (1 - \lambda z)^{-1} \left(L(\lambda) - \frac{1}{2}(z\mathbf{s} + \mathbf{s}z) \right), \quad z = g \tilde{z} g^T. \quad (2.31)$$

The corresponding r -matrix is dynamical.

So, for initial Hamiltonian H (2.7) we know two Lax matrices $L(\lambda)$ (2.4) and $\tilde{L}(\lambda)$ (2.23) associated with the physically different coordinate systems. For these Lax matrix we constructed by two perturbations L_y (2.11), L_z (2.31) in the rest frame and \tilde{L}_y (2.25), \tilde{L}_z (2.29) in the body frame.

3 Further examples

According to [1], we shall try to exclude the nonphysical degrees of freedom using symmetry of the Hamiltonian H (2.7) and its perturbations $H_{1,2}$ (2.19, 2.20). Assume that $A = E$ is the truncated identity matrix $E_{ij} = \delta_{ij}$.

Below all the Hamiltonians will be expressed through kinetic momentum $\ell_{ij} \in so(p)$ and the entries of the Poisson vectors $x_i = k^T f_i$, where f_i are the column vectors of the matrix F . These variables are canonical coordinates on the algebra $e(p, q)^* = so(p) \ltimes (R^p \otimes \mathbb{R}^q)$ [1].

3.1 Algebra $so(p, 1)$, Lagrange top and spherical pendulum.

If $q = 1$ the phase space is $T^*SO(p)$. We can put

$$m = 0, \quad r = 1, \quad (3.1)$$

without loss of generality. In this particular case the deformations (2.19) and (2.20) of the initial Hamiltonian are quadratic polynomials instead of cubic ones.

Following to [1], we can rewrite Hamiltonian (2.7) in the form

$$H = \frac{1}{2} \sum_{i,j=1}^p \ell_{ij}^2 + \frac{\gamma}{2} \sum_{i,j=2}^p \ell_{ij}^2 - (e_1, x), \quad x = k^T f. \quad (3.2)$$

Here e_1 is a first vector of the standard basis in \mathbb{R}^p .

The Hamiltonian (3.2) describes rotation of a rigid body around a fixed point in a homogeneous gravity field. The vector x is the vector along the gravity field, with respect to the body frame and e_1 is the vector pointing from the fixed point to the center of mass of the body.

It is Lagrange case because the body is rotationally symmetric and the fixed point lies on the symmetry axis e_1 . The perturbations (2.19) and (2.20) of the Hamiltonian (3.2)

$$H_1 = H + \frac{c^2}{2} (e_1, k^T f)^2 = H + \frac{c^2}{2} (e_1, x)^2$$

$$H_2 = H + c (\ell e_1, k^T f) = H + c (\ell e_1, x).$$

are quadratic polynomials and, therefore, the corresponding equations of motion have the form of the Kirchhoff equations.

Let $x = k^T f$ and π are canonical coordinates in \mathbb{R}^{2p} , $\{\pi_i, x_j\} = \delta_{ij}$. Substituting $\ell_{i,j} = x_i \pi_j - \pi_i x_j$ in $L(\lambda)$ (2.4) one gets a Lax matrix for the spherical pendulum in appropriate physical coordinates [1]. The corresponding Hamiltonian

$$H = \frac{1}{2} \sum \pi_i^2 - \sum A_i x_i.$$

describes a motion on the sphere S^{p-1} , $(x, x) = 1$, $(x, \pi) = 0$, in a homogeneous gravity field. Its deformations (2.20) and (2.19) look like

$$H_1 = H + c \sum A_i \pi_i, \quad H_2 = H + \frac{c^2}{2} \left(\sum A_i x_i \right)^2.$$

3.2 The Kowalevski top.

According [1, 5], the generalized p -dimensional Kowalevski top is the reduced system (2.7) with respect to the action of the subgroup $SO(q)$. The reduction amounts to imposing the constraints

$$m + P \ell P = 0, \quad r = 1. \quad (3.3)$$

Here P means the orthogonal projection from \mathbb{R}^p onto subspace of \mathbb{R}^q spanned by first q vectors of the standard basis. The reduced phase space is $T^*SO(p)$ with its canonical Poisson structure.

Inserting the constraints (3.3) into (2.4) one gets the Lax matrix for the generalized Kowalevski top

$$L^{Kow}(\lambda) = \begin{pmatrix} 0 & k E \\ E^T k^T & 0 \end{pmatrix} \lambda - \begin{pmatrix} k \ell k^T & 0 \\ 0 & -E^T \ell E \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix}. \quad (3.4)$$

Proposition 4 *The spectral invariants of the Lax matrices $L^{Kow}(\lambda)$ (3.4) and $L_y(\lambda)$ (2.11) by (3.3) are in the involution with respect to Lie-Poisson brackets on $T^*SO(p)$.*

We present a direct proof of the involutivity of invariants without recourse to Hamiltonian reduction.

In the rest frame it can easily be shown that the reduced Lax matrices satisfy equation (2.5) with the same kernels $r_{12}(\lambda, \mu)$ (2.6) and $r_{12}^y(\lambda, \mu)$ (2.13). These kernels do not change by the Poisson reduction (3.3) and the corresponding operators R (1.1) and R_y (1.9) remain differences of the same projectors.

It should be pointed out that constraints (3.3) agree with the Euler-Lagrange description of the top, but disagree with its Euler-Poisson description [1]. In the body frame the Lax matrix is given by

$$\tilde{L}^{Kow}(\lambda) = \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix} \lambda - \begin{pmatrix} \ell & 0 \\ 0 & -E^T \ell E \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & k^T F \\ F^T k & 0 \end{pmatrix}. \quad (3.5)$$

Proposition 5 *The spectral invariants of the Lax matrices $\tilde{L}^{Kow}(\lambda)$ (3.5) and $\tilde{L}_z(\lambda)$ (2.29) by (3.3) are in the involution with respect to Lie-Poisson brackets on $T^*SO(p)$.*

In the body frame reduction (3.3) does not Poisson mapping which changes R -brackets. Thus, after reduction the Lax matrix $\tilde{L}(\lambda)$ (2.23) satisfies (2.5) with the reduced kernel

$$\tilde{r}_{12}^{Kow}(\lambda, \mu) = \tilde{r}_{12}(\lambda, \mu) - \sum_{\alpha=1}^{\frac{p(p-1)+q(q-1)}{2}} Z_\alpha \otimes \hat{P} Z_\alpha \hat{P}. \quad (3.6)$$

Here

$$\hat{P} = \begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix}, \quad \hat{P} Z_\alpha \hat{P} = \begin{cases} Z_{\alpha + \frac{p(p-1)}{2}}, & 0 < \alpha \leq \frac{q(q-1)}{2}; \\ 0, & \frac{q(q-1)}{2} < \alpha \leq \frac{p(p-1)}{2}; \\ Z_\alpha, & \frac{p(p-1)}{2} < \alpha. \end{cases}$$

Algebro-geometric description of the corresponding "non-standard" R operator may found in [7].

After reduction (3.3) the perturbed Lax matrix $\tilde{L}_z(\lambda)$ (2.29) satisfies (2.5) with the reduced r -matrix

$$\tilde{r}_{12}^{zKow}(\lambda, \mu) = \left[1 \otimes (1 - \lambda \tilde{z})^{-1} \right] \tilde{r}_{12}^{Kow}(\lambda, \mu) \left[(1 - \mu \tilde{z}) \otimes 1 \right].$$

Substituting constraints (3.3) into (2.7) gives the reduced Hamiltonian on $T^*SO(p)$

$$H = \frac{1}{4} \left(\sum_{i,j=1}^p \ell_{ij}^2 + \sum_{i,j=1}^q \ell_{ij}^2 \right) - \sum_{i=1}^q (e_i, k^T f_i), \quad (3.7)$$

where e_1, \dots, e_p is the moving orthonormal frame associated with the top, and f_1, \dots, f_q are the column vectors of F . This Hamiltonian describes a p -dimensional top with a very special $SO(q) \times SO(p-q)$ -symmetric inertial tensor, rotating under the influence of q constant forces [1].

Inserting constraints (3.3) into H_1 (2.19) and H_2 (2.20) one gets quadratic polynomials

$$H_1 = H - c \sum_{i=1}^q (P \ell P e_i, k^T f_i) + \frac{c^2}{2} \sum_{i=1}^q (k^T f_i, k^T f_i), \quad (3.8)$$

and

$$H_2 = H - c \sum_{i=1}^q (\ell e_i, k^T f_i), \quad (3.9)$$

It means that the corresponding equations of motion have the form of the Kirchhoff equations of motion of a rigid body in the ideal fluid.

Thus, the characteristic doubling of the terms ℓ_{ij}^2 , for $i, j \leq q$ in (3.7), and existence of the nontrivial quadratic perturbations H_1^y (3.8) and H_2^y (3.9) are results of reduction with respect to $SO(q)$.

The case $p = 3$, $q = 2$ and $f_2 = 0$ gives usual Kowalevski top on the algebra $e(3)$ with a Poisson vector $x = k^T f_1$. The deformations (2.19-2.20) of the standard Hamiltonian became

$$H_1 = \frac{1}{2} \left(\ell_{23}^2 + \ell_{13}^2 + 2\ell_{12}^2 \right) - x_1 + c \ell_{12} x_2 + \frac{c^2}{2} (x_1^2 + x_2^2). \quad (3.10)$$

and

$$H_2 = \frac{1}{2} \left(\ell_{23}^2 + \ell_{13}^2 + 2\ell_{12}^2 \right) - x_1 + c (\ell_{12} x_2 + \ell_{13} x_3). \quad (3.11)$$

Recall, H_1 and H_2 are two different forms of the one Hamiltonian H^y , which are connected by canonical transformation (2.16).

3.3 Integrable systems on $so(4)$.

If $p = q = 3$ and $F_{ij} = 0$ the Hamiltonian H (2.7) describes integrable system on the phase space $so(4) = so(3) \oplus so(3)$. Let

$$u_k = -\varepsilon_{ijk} \ell_{jk}, \quad v_k = -\varepsilon_{ijk} m_{jk},$$

where ε_{ijk} is the totally skew-symmetric tensor. The Hamiltonian (2.7) becomes

$$H = \sum_{k=1}^3 \alpha_k (u_k^2 + v_k^2) - 2 \sum_{k=1}^3 \beta_k u_k v_k.$$

The coefficients

$$\alpha_k = \varepsilon_{ijk}^2 \frac{a_i b_i - a_j b_j}{a_i^2 - a_j^2}, \quad \beta_k = \varepsilon_{ijk}^2 \frac{a_i b_j - a_j b_i}{a_i^2 - a_j^2}$$

satisfy the relation

$$(\alpha_1 - \alpha_2)\beta_3^2 + (\alpha_2 - \alpha_3)\beta_1^2 + (\alpha_3 - \alpha_1)\beta_2^2 + (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) = 0.$$

Thus, according to [8], this integrable system belongs to the so-called Steklov-Manakov family of integrable systems, characterized by the property that there exists an additional quadratic integral. In this case $F = 0$ and proposed deformations (2.19-2.20) are trivial.

The more complicated passage to the algebra $so(4)$ was proposed in [4]. Let us apply composition of the gauge transformations to the Lax matrix $L_y^{(2)}$ (2.22)

$$\hat{L}_y = (1 - \lambda^{-1}g) \left(g^T L_y^{(2)}(\lambda) g \right) (1 - \lambda^{-1}g)^{-1}, \quad g = c \begin{pmatrix} 0 & k^T F \\ 0 & 0 \end{pmatrix}.$$

Substituting constraints (3.3) one gets

$$\hat{L}_y = \left[1 + \frac{c^2}{\lambda^2} \begin{pmatrix} 0 & 0 \\ 0 & F^T F \end{pmatrix} \right]^{-1} \times \quad (3.12)$$

$$\left[\begin{pmatrix} 0 & E \\ E^T & 0 \end{pmatrix} \lambda - \begin{pmatrix} \ell & 0 \\ 0 & -E^T \ell E \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & \gamma^T F \\ F^T \gamma & 0 \end{pmatrix} + \frac{c^2}{\lambda^2} \begin{pmatrix} 0 & 0 \\ 0 & F^T d F \end{pmatrix} \right],$$

where

$$\gamma = (1 + \ell^T) k \quad \text{and} \quad d = k^T \ell k. \quad (3.13)$$

The matrix

$$h = \begin{pmatrix} 0 & 0 \\ 0 & F^T F \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & r^T F^T k k^T F r \end{pmatrix}$$

is numerical because $r = 1$ (3.3) only.

Proposition 6 *The Lax matrix $\hat{L}_y(\lambda)$ (3.12) satisfies (2.5) with numerical r -matrix*

$$\hat{r}_{12}^{Kow}(\lambda, \mu) = \left[1 \otimes \left(1 + \frac{c^2}{\mu^2} h \right)^{-1} \right] \tilde{r}_{12}^{Kow}(\lambda, \mu) \left[\left(1 + \frac{c^2}{\lambda^2} h \right) \otimes 1 \right]. \quad (3.14)$$

The proof is straightforward. Thus, for the generalized Kowalevski top we constructed third Lax matrix with the numerical r -matrix. In contrast with the previous two Lax matrices in this case we do not know the physical meaning of the corresponding coordinate system.

The non-zero Poisson brackets between variables ℓ, γ and d (3.13) are

$$\begin{aligned}
\{\ell_{ij}, \ell_{mn}\} &= \delta_{in} \ell_{jm} + \delta_{jm} \ell_{in} - \delta_{im} \ell_{jn} - \delta_{jn} \ell_{im}, \\
\{\ell_{ij}, \gamma_{nm}\} &= \delta_{jm} \gamma_{ni} - \delta_{im} \gamma_{nj}, \\
\{d_{ij}, d_{kn}\} &= \delta_{ik} d_{jn} + \delta_{jn} d_{ik} - \delta_{in} d_{jk} - \delta_{jk} d_{in} \\
\{d_{ij}, \gamma_{nk}\} &= \delta_{in} \gamma_{kj} - \delta_{jn} \gamma_{ki}, \\
\{\gamma_{ij}, \gamma_{kn}\} &= c^2 (\delta_{jn} \ell_{is} - \delta_{ik} d_{jn}).
\end{aligned} \tag{3.15}$$

Entries of matrix ℓ jointly with entries of any column of γ form subalgebras with respect to bracket (3.15). If $\text{rank } F = 1$, then we can put

$$F^T dF = 0, \quad (\gamma^T F)_{ij} = \delta_{j1} \gamma_{i1}$$

and, therefore, the Lax matrix (3.12) is defined on the one of such subalgebras only.

Let $p = 3$, in canonical variables

$$J_i = -\varepsilon_{ijk} \ell_{jk}, \quad y_j = \gamma_{j1}$$

the brackets (3.15) coincide with the standard Lie-Poisson brackets on $so(4)$

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, y_j\} = \varepsilon_{ijk} y_k, \quad \{y_i, y_j\} = c^2 \varepsilon_{ijk} J_k. \tag{3.16}$$

Thus, subalgebra $\{\ell, \gamma_{j1}\}$ is isomorphic to the Lie algebra $so(4)$ [4].

If $q = 1$ or $q = 3$ the perturbed Hamiltonian (2.20)

$$H_2 = J_1^2 + J_2^2 + J_3^2 - y_1$$

describes the Lagrange top on the algebra $so(4)$.

For $q = 2$ the perturbed Hamiltonian (2.20)

$$H_2 = \frac{1}{2} (J_1^2 + J_2^2 + 2J_3^2) - y_1.$$

describes the Kowalevski top on the Lie algebra $so(4)$. As for the usual Kowalevski top [1], we can generalize constraints (3.3) and describe the Kowalevski gyrostat on $so(4)$ [4].

The Lax matrices for the Lagrange and Kowalevski tops on $so(4)$ were constructed in [4]. In this section we calculate the corresponding r -matrix (3.14) and propose generalization on the case $p \neq 3$.

4 Summary

We construct the Lax matrices for an integrable deformation of the known integrable system of two interacting p -dimensional and q -dimensional spherical tops. One gets r -matrices associated with this model and with some reduced systems such as generalized Lagrange and Kowalevski tops.

The matrix y (2.9), which determines perturbation (1.11) of the known Lax matrix, is the solution of the equations (2.10). Relation of such equations on the associative algebras with a general theory of the classical R -matrices on the twisted loop algebras remains an open question. Some another constructions of such deformations may be found in [2].

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